DEPARTMENT OF EDUCATION

CENTRAL PROVINCE

G.C.E.(A/L) Examination – 2023

Combined Mathematics I

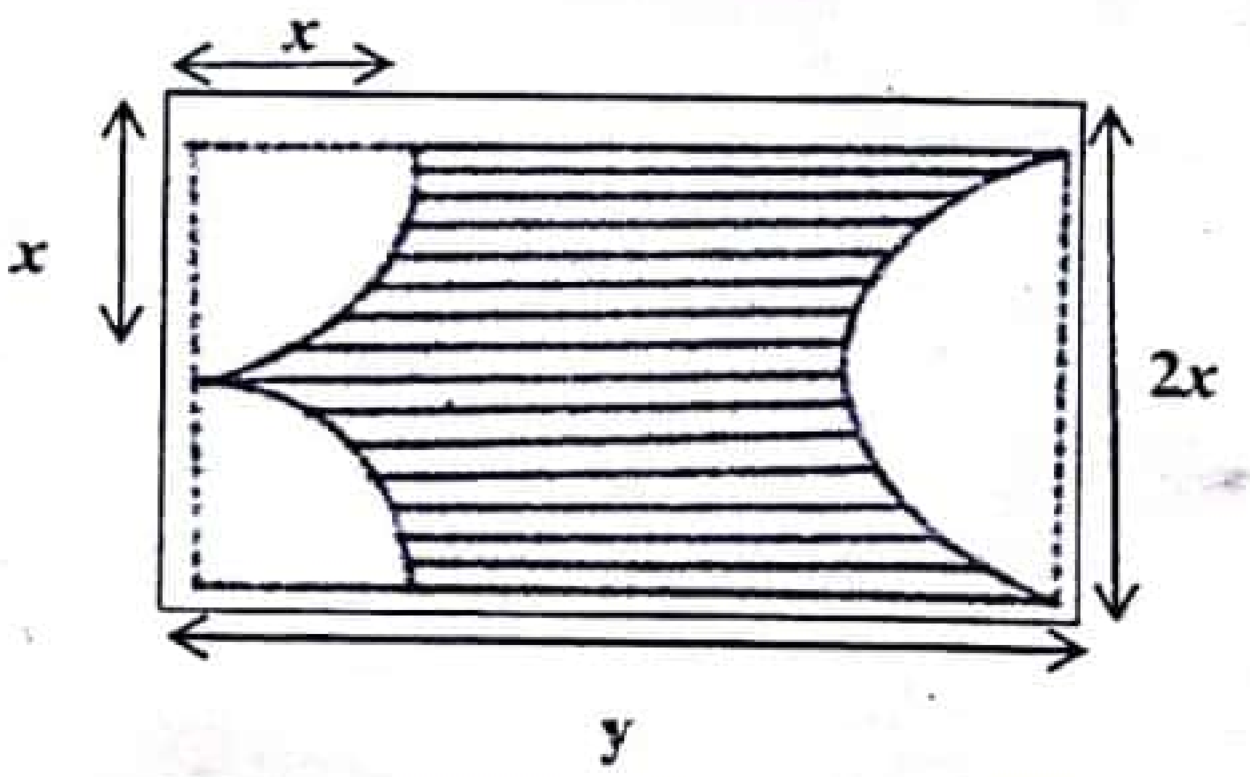
Marking Scheme

Grade 13

$$A = \begin{pmatrix} 1 & 1 \\ p & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_1 : x^2 + y^2 + \frac{15}{2}x - \frac{15}{2}y + 26 = 0$$

$$\sum_{r=1}^{n} u_r = \frac{7}{12} - \frac{2n^2 + 7n + 7}{2(n+1)(n+2)(n+3)}$$



$$|z + 1 - 2i| \le 2,\ Agr(z+3) < \frac{\pi}{4}$$

$$\lim_{x \to 0} \frac{\sin \frac{\pi}{2}(1 + \cos 2x)}{\sqrt{1 - x^2} - 1} = 2\pi$$

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$\tan 5\theta = \frac{\tan\theta\left[\tan^4\theta - 10\tan^2\theta + 5\right]}{5\tan^4\theta - 10\tan^2\theta + 1}$$

$$I = \int_0^{\pi} \frac{x\cos 2x}{1 + \cos 2x}dx = \frac{\pi^2}{2}$$

$$\cot^{-1}(x+2) + \tan^{-1}(x^2+1) = \frac{\pi}{2}$$

$$f''(x) = \frac{3(x+3)(4x+1)}{2(x-1)^5}$$

01. $1\cdot5 + 2\cdot5^2 + 3\cdot5^3 + \cdots + n\cdot5^n = \dfrac{(4n-1)\cdot5^{n+1}+5}{16}$

When $n=1$

$LHS = 1\cdot5 = 5$

$RHS = \dfrac{(4-1)\cdot5^2+5}{16} = \dfrac{80}{16} = 5$

$\therefore LHS = RHS$

the result is true for $n=1$. (5)

Let assumed that the result is true for $n=p$ (5)

$$1\cdot5 + 2\cdot5^2 + 3\cdot5^3 + \cdots + p\cdot5^p = \frac{(4p-1)\cdot5^{p+1}+5}{16}$$

To have to prove

the result is true for $n=p+1$

i.e. $1\cdot5 + 2\cdot5^2 + 3\cdot5^3 + \cdots + (p+1)\cdot5^{p+1} = \dfrac{[4(p+1)-1]\cdot5^{p+2}+5}{16}$

Proof

Add $(p+1)^{th}$ term in the both side of the assumption

$$1\cdot5 + 2\cdot5^2 + 3\cdot5^3 + \cdots + p\cdot5^p = \frac{(4p-1)\cdot5^{p+1}+5}{16} + (p+1)\cdot5^{p+1} \quad (5)$$

$$= \frac{[4p-1+16(p+1)]\cdot5^{p+1}+5}{16}$$

$$= \frac{(20p+15)\cdot5^{p+1}+5}{16}$$

$$= \frac{(4p+3)\cdot5^{p+2}+5}{16}$$

$$= \frac{[4(p+1)-1]\cdot5^{p+2}+5}{16} \quad (5)$$

According to the principle of mathematical induction the result is true for all possitive integer n. (5)

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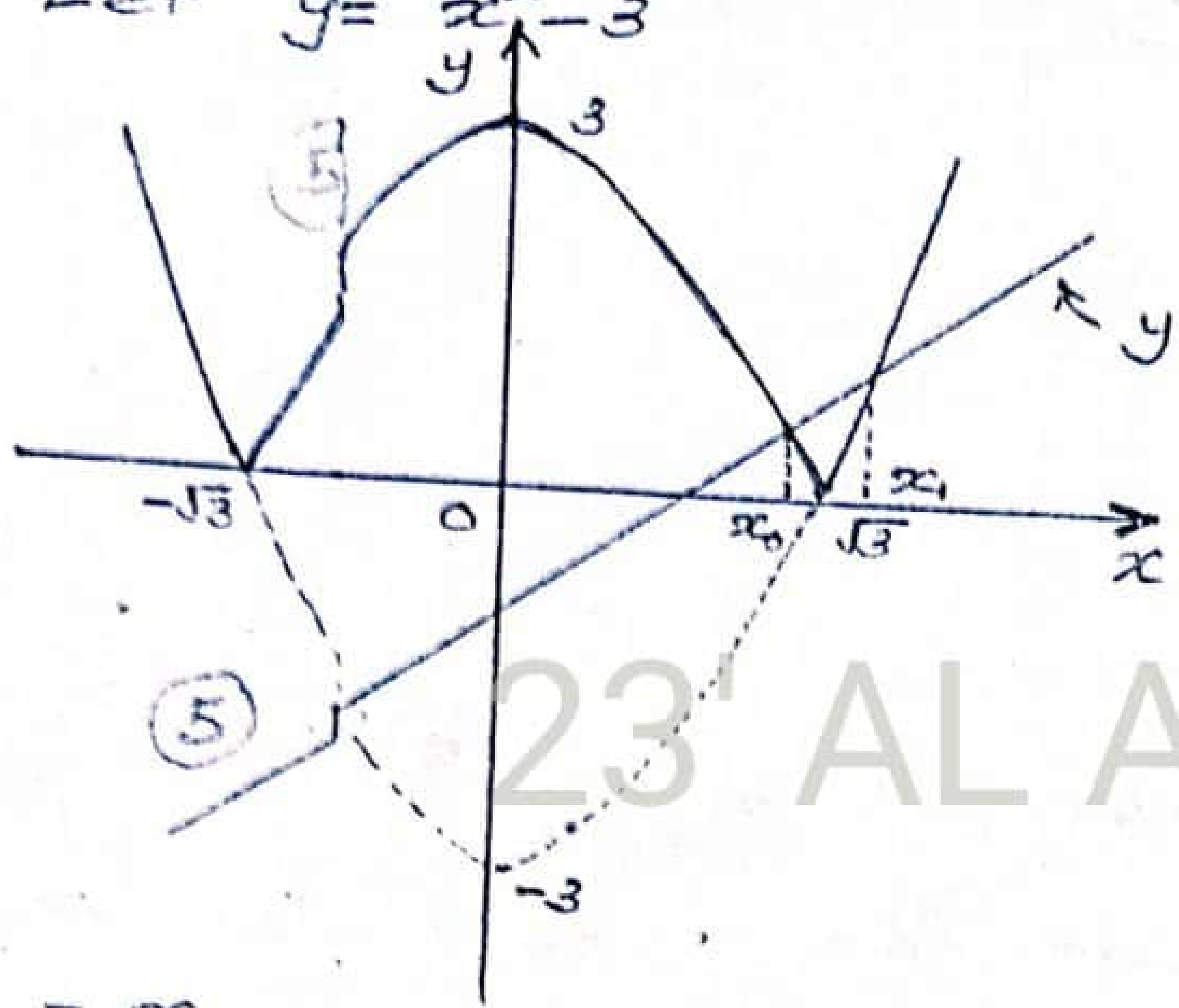
02. Let $y = x^2 - 3$



$$\left.\begin{aligned} y_0 &= x_0 - 1 \\ y_0 &= -x_0^2 + 3 \end{aligned}\right\}$$

$$x_0 - 1 = -x_0^2 + 3$$

$$x_0^2 + x_0 - 4 = 0$$

$$\left(x_0 + \tfrac{1}{2}\right)^2 - 4 - \tfrac{1}{4} = 0$$

$$x_0 = -\tfrac{1}{2} + \tfrac{\sqrt{17}}{2} \qquad x_0 = -\tfrac{1}{2} - \tfrac{\sqrt{17}}{2} \quad 5$$

$$\left.\begin{aligned} y_1 &= x_1 - 1 \\ y_1 &= x_1^2 - 3 \end{aligned}\right\}$$

$$x_1 - 1 = x_1^2 - 3$$

$$x_1^2 - x_1 - 2 = 0$$

$$(x_1 - 2)(x_1 + 1) = 0$$

$$x_1 = 2 \qquad x_1 = -1 \quad 5$$

Required range

$$\frac{\sqrt{17} - 1}{2} < x < 2 \quad 5$$

$\boxed{25}$

03. $|z + 1 - 2i| \leq 2$

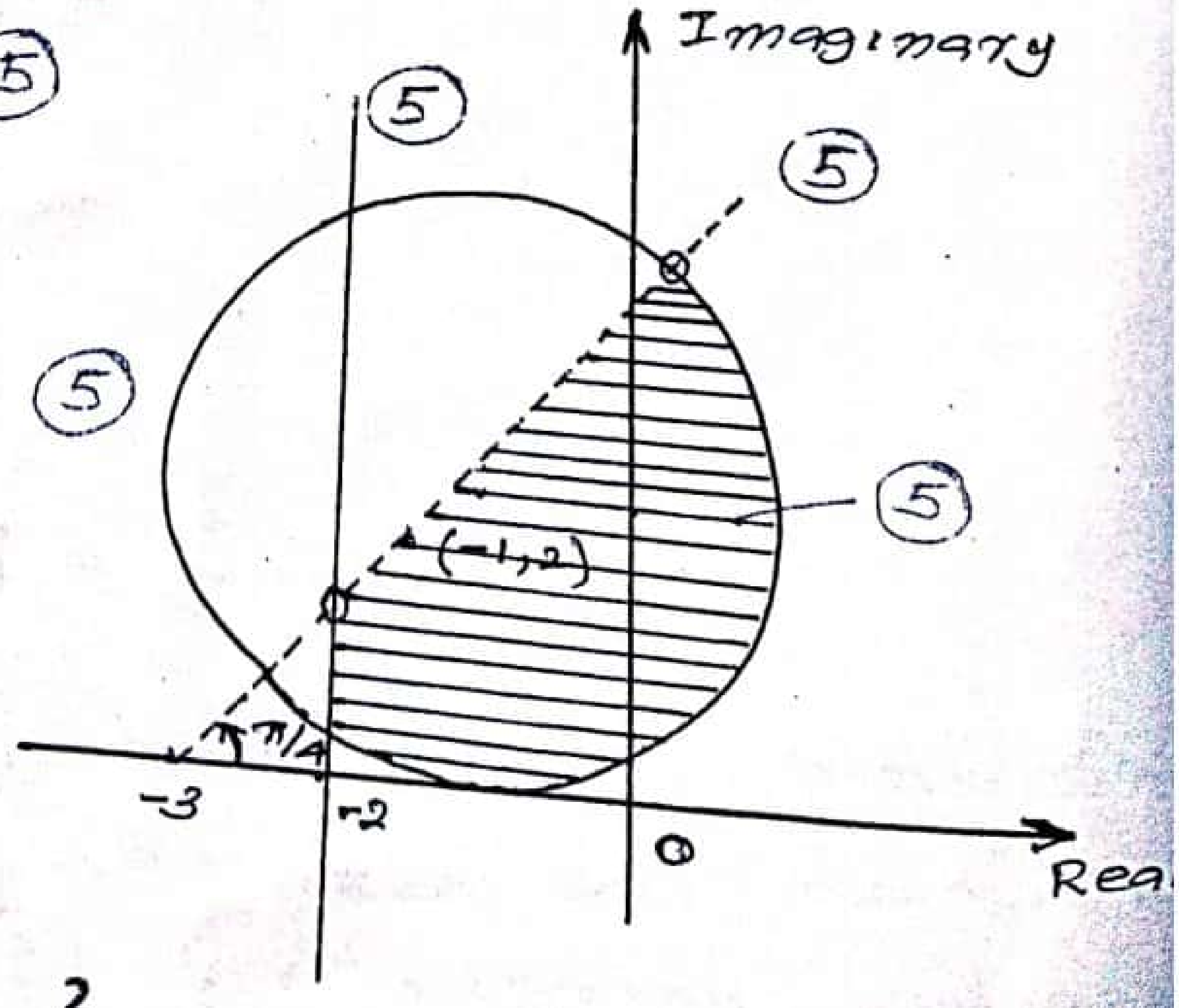
$$|(x+1) + i(y-2)| \leq 2$$

$$\sqrt{(x+1)^2 + (y-2)^2} \leq 2 \quad 5$$

$$(x+1)^2 + (y-2)^2 \leq 2^2$$

$$\text{Arg}(z+3) < \frac{\pi}{4}$$

$$\text{Re}(z) > -2$$



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04. $(3a+2x)^4(ax^2+2)$

$$= \left[{}^4C_0(3a)^4 + {}^4C_1(3a)^3(2x) + {}^4C_2(3a)^2(2x)^2 + {}^4C_3(3a)(2x)^3 + {}^4C_4(2x)^4\right](ax^2+2)$$ 5

coefficient of x^4 5 5

$${}^4C_2(3a)^2 \cdot 2^2 \cdot a + {}^4C_4 \cdot 2^4 \cdot 2 = -93$$ 5

$$\frac{4!}{2!\,2!}\, 9a^3 \cdot 4 + 32 = -93$$

$$6 \times 36a^3 = -125$$

$$a^3 = \left(-\frac{5}{6}\right)^3$$

$$a = -\frac{5}{6}$$ 5

$\boxed{25}$

05. $$\lim_{x\to 0} \frac{\sin\frac{\pi}{2}(1+\cos 2x)}{\sqrt{1+x^2}-1}$$

$$= \lim_{x\to 0} \frac{\sin\frac{\pi}{2}(1+2\cos^2 x-1)\left[\sqrt{1+x^2}+1\right]}{1+x^2-1}$$ 5 5

$$= \lim_{x\to 0} \frac{\sin\pi\cos^2 x}{x^2} \cdot \left[\sqrt{1+0}+1\right]$$

$$= 2\lim_{x\to 0} \frac{\sin(\pi-\pi\cos^2 x)}{x^2}$$

$$= 2\lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{x^2}$$ 5

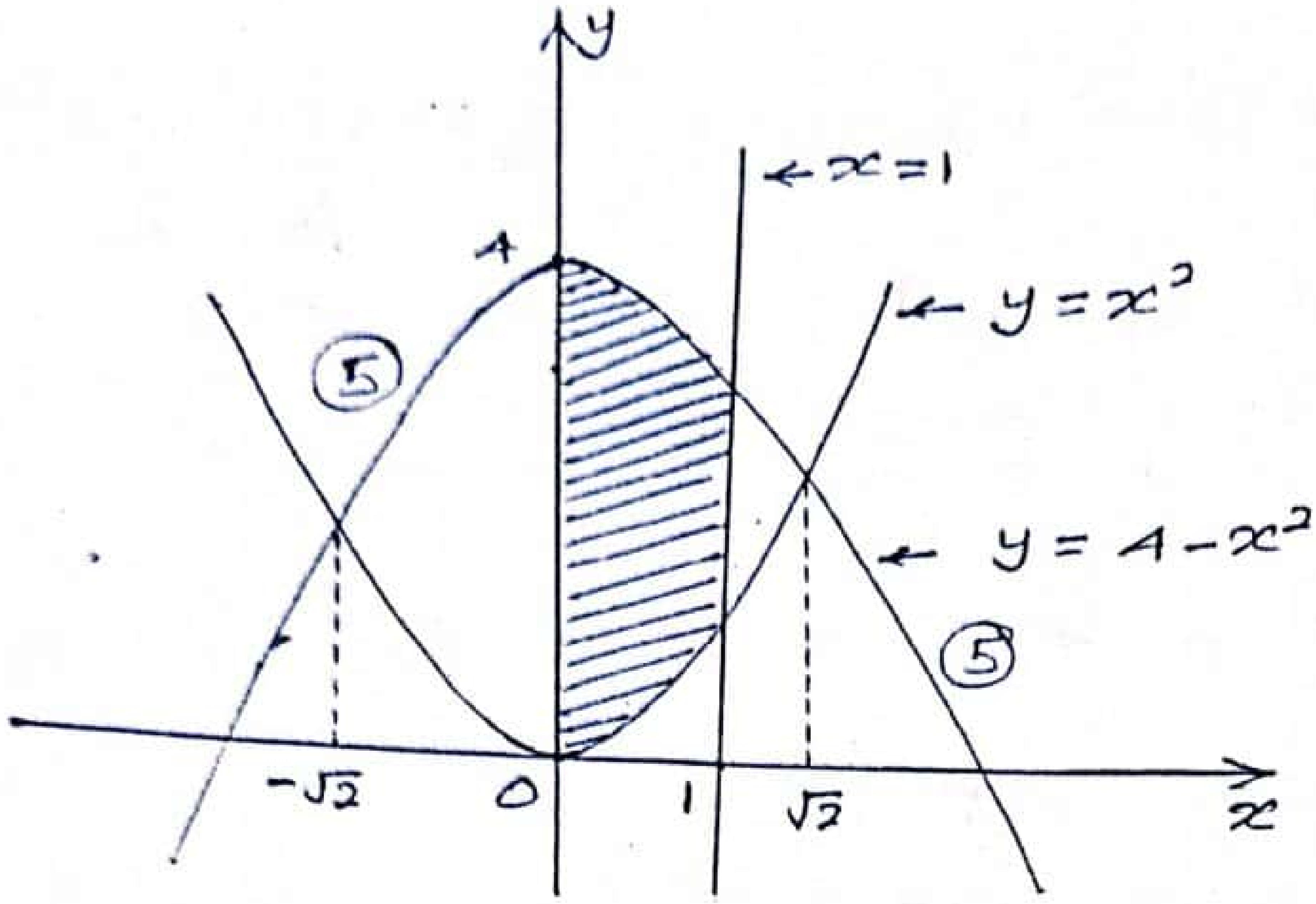
$$= 2\lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x} \cdot \pi\lim_{x\to 0} \frac{\sin^2 x}{x^2}$$

$$= 2\pi \underbrace{\lim_{\pi\sin^2 x\to 0} \frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x}}_{1} \cdot \underbrace{\left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2}_{1}$$ 5

$$= 2\pi$$ 5

$\boxed{25}$

06.



Area enclosed by curves

$$A = \int_0^1 (4-x^2)\,dx - \int_0^1 x^2\,dx$$

$$= \left[4x - \frac{x^3}{3}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1$$

$$= 4 - \frac{1}{3} - \frac{1}{3} = \frac{10}{3} \text{ sq units} \quad 5$$

Volume enclosed by curves

$$V = \int_0^1 \pi (4-x^2)^2\,dx - \int_0^1 \pi x^4\,dx \quad 5$$

$$= \pi \left\{ \int_0^1 (16 - 8x^2 + x^4 - x^4)\,dx \right\}$$

$$= 8\pi \int_0^1 (2 - x^2)\,dx$$

$$= 8\pi \left[2x - \frac{x^3}{3}\right]_0^1$$

$$= 8\pi \left[2 - \frac{1}{3}\right]$$

$$= \frac{40\pi}{3} \text{ cubic unit} \quad 5$$

$\boxed{25}$

7. $x = t + \sqrt{t}$ $\qquad\qquad y = t - \sqrt{t}$

$$\frac{dx}{dt} = 1 + \frac{1}{2\sqrt{t}} \qquad\qquad \frac{dy}{dt} = 1 - \frac{1}{2\sqrt{t}}$$

$$= \frac{2\sqrt{t}+1}{2\sqrt{t}} \quad (5) \qquad\qquad = \frac{2\sqrt{t}-1}{2\sqrt{t}}$$

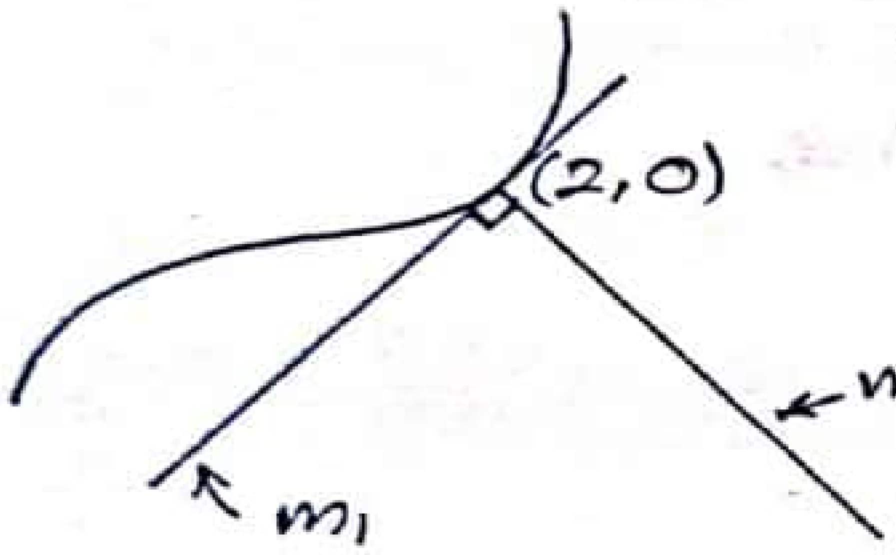
chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \quad (5)$$

$$= \frac{2\sqrt{t}-1}{2\sqrt{t}} \cdot \frac{2\sqrt{t}}{2\sqrt{t}+1} = \frac{2\sqrt{t}-1}{2\sqrt{t}+1}$$

The gradient of the tangent drawn at $t=1$

$$m_1 = \left.\frac{dy}{dx}\right|_{t=1} = \frac{2\sqrt{1}-1}{2\sqrt{1}+1} = \frac{1}{3} \quad (5)$$



The eq^n of the normal

$$\frac{y-0}{x-2} = -3$$

$$3x + y - 6 = 0 \quad (5)$$

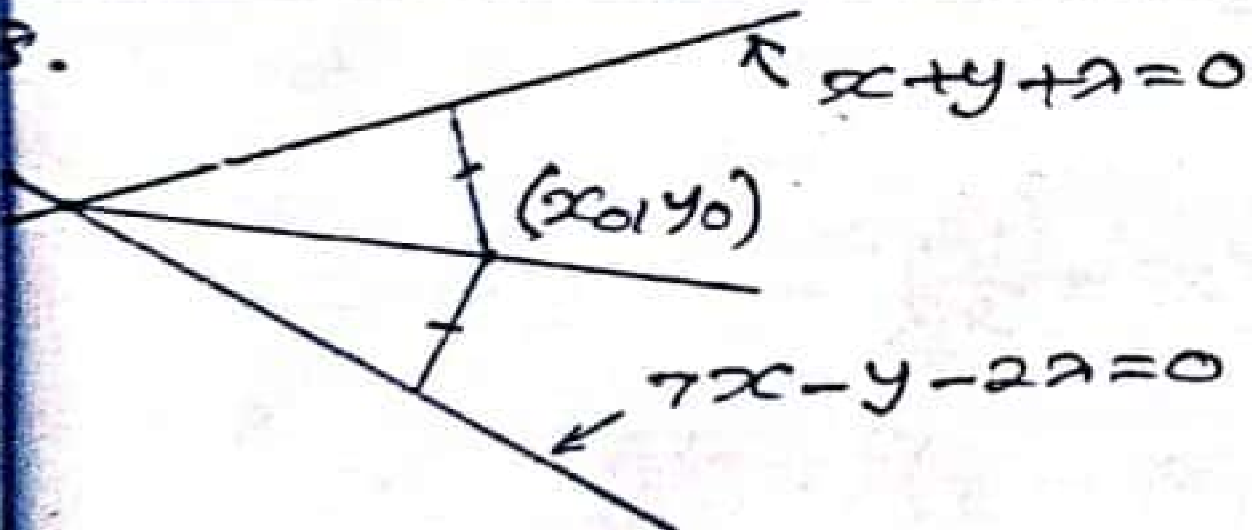
$(2a+1, a-2)$ satisfy the eq^n

$$3(2a+1) + a - 2 - 6 = 0$$

$$a = \frac{5}{7} \quad (5)$$

$\boxed{25}$

8.



By equating ⊥^{r} distance

$$(5) \quad \frac{|x_0+y_0+a|}{\sqrt{2}} = \frac{|7x_0-y_0-2a|}{5\sqrt{2}}$$

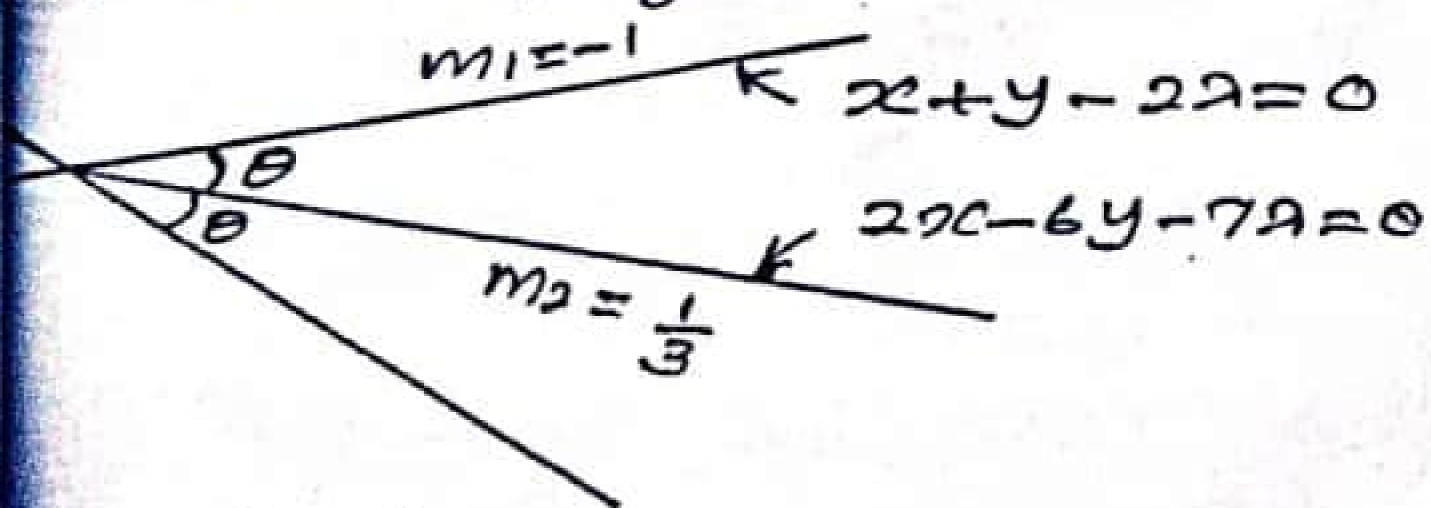
The eq^n of bisectors

$$5(x+y+a) = \pm(7x-y-2a)$$

$$\Rightarrow \quad 2x-6y-7a=0$$

$$12x+4y+3a=0 \quad (5)$$

Let assume that $2x-6y-7a=0$ be the acute angle bisector



$$\tan\theta = \left|\frac{m_2-m_1}{1+m_1m_2}\right| \quad (5)$$

$$= \left|\frac{\frac{1}{3}+1}{1-\frac{1}{3}}\right| = 2 > 1$$

$2\theta < 90^\circ$

$\theta < 45^\circ \quad (5)$

$\tan\theta < \tan 45^\circ$

$\tan\theta < 1$

the eq^n of acute angle bisector

$$12x+4y+3a=0$$

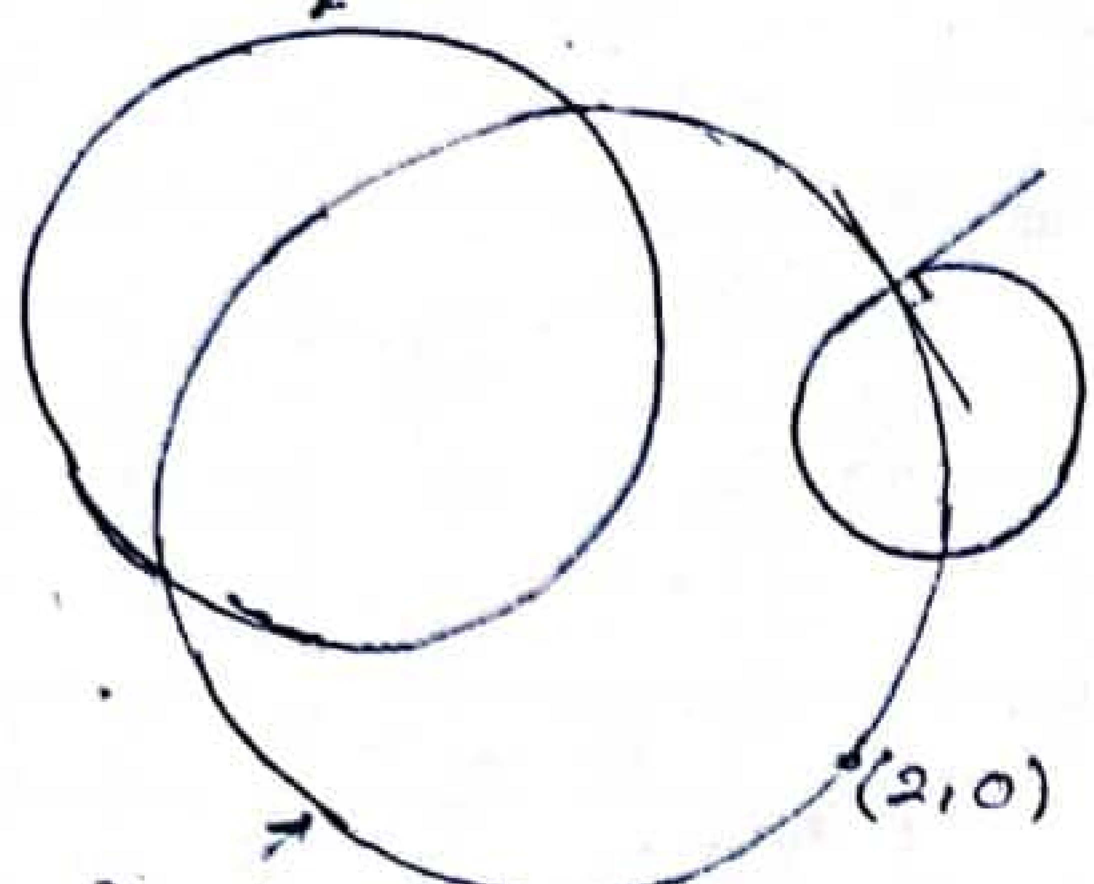
$(1,-2)$ satisfied $\quad (5)$

$$12-8+3a=0$$

$$a = -\frac{4}{3}$$

$\boxed{25}$

09. $S_1 : x^2+y^2-1=0$



$S: x^2+y^2+2gx+2fy+c=0$

$S=0$ and $S_2=0$ intersect orthogonally

$2g_1g_2+2f_1f_2 = c_1+c_2$ (5)

$2.g.0 + 2.f.(-2) = c-5$

$-4f = -1-5$

$f = \frac{3}{2}$

$(2,0)$ satisfy $S=0$

$4+0+4g+0+c=0$

(5) $4g+c=-4$ — (1)

For the eq^n of common chord

$S-S_1$

(5) $2gx+2fy+c+1=0$

the centre of $S_1=0$ $(0,0)$ satisfy the eq

$0+0+c+1=0$

$c=-1$

(1) $\Rightarrow$ $4g-1=-4$

(5) $4g=-3$

$g=-\frac{3}{4}$

$x^2+y^2+2\left(-\frac{3}{4}\right)x+2\cdot\frac{3}{2}y-1=0$

$2x^2+2y^2-3x+6y-2=0$ (5)

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10. $\underbrace{\cot^{-1}(x+2)}_{\alpha} + \underbrace{\tan^{-1}(x^2+1)}_{\beta} = \frac{\pi}{2}$

$\alpha+\beta = \frac{\pi}{2}$

$\alpha = \frac{\pi}{2}-\beta$

$\tan\alpha = \tan\left(\frac{\pi}{2}-\beta\right)$

$= \cot\beta$ (5)

$= \frac{1}{\tan\beta}$

$\tan\alpha\tan\beta = 1$

$\frac{1}{x+2}\cdot(x^2+1) = 1$ (5)

$x^2-x-1=0$ (5)

$\left(x+\frac{1}{2}\right)^2-1-\frac{1}{4}=0$

$\left(x-\frac{1}{2}\right)^2 = \frac{5}{4}$

$x-\frac{1}{2} = \pm\frac{\sqrt{5}}{2}$

$x=\frac{1}{2}+\frac{\sqrt{5}}{2}$ (5) or $x=\frac{1}{2}-$ (5)

25

11. (a) $f(x) = ax^2 + bx + c \quad ; \quad a \neq 0$

$$= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] \quad 5$$

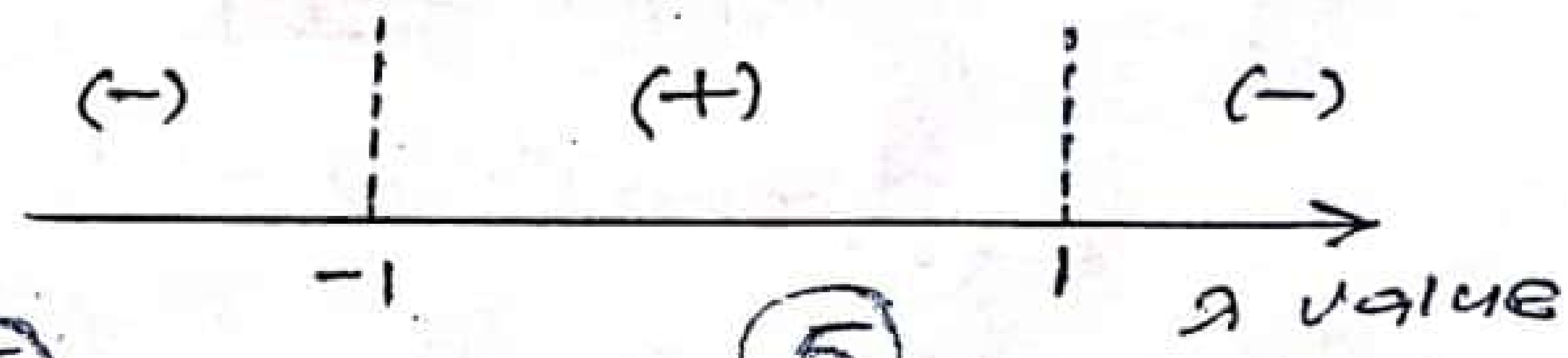
For $f(x) > 0$

$a > 0$ and $4ac - b^2 > 0$ 5 $\boxed{10}$

$\Rightarrow a > 0$ and $b^2 - 4ac < 0$

$f(x) = x^2 + 2x + \lambda^2$

$4 - 4 \cdot 1 \cdot \lambda^2 < 0$ 5



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$F(\lambda) = (1-\lambda)(1+\lambda) < 0$

when $\lambda < -1$
$F(\lambda) = (+)(-)$
$= (-)$

when $-1 < \lambda < 1$
$F(\lambda) = (+)(+)$
$= (+)$

when $\lambda > 1$
$F(\lambda) = (-)(+)$
$= (-)$

The set of λ, such that $f(x) > 0$ is given by

$\lambda \in (-\infty, -1) \cup (1, \infty)$ 10 $\boxed{20}$

$x^2 + 2x + \lambda^2 = 0 \qquad K - 2\alpha, \quad K - 2\beta$

sum of roots

$K - 2\alpha + K - 2\beta = -2$ 5

$2K - 2(\alpha + \beta) = -2$

$\alpha + \beta = K + 1$ 5

product of roots 5

$(K - 2\alpha)(K - 2\beta) = \lambda^2$

$K^2 - 2K(\alpha + \beta) + 4\alpha\beta = \lambda^2$

$K^2 - 2K(K+1) + 4\alpha\beta = \lambda^2$ 5

$\alpha\beta = \dfrac{\lambda^2 + K^2 + 2K}{4}$

The eq^n of root α and β

$(x - \alpha)(x - \beta) = 0$ 5

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$x^2 - (K+1)x + \dfrac{\lambda^2 + K^2 + 2K}{4} = 0,$ 5 $\boxed{30}$

$g(x) = 0$

$$g(x) = x^2 - (k+1)x + \frac{a^2+k^2+2k}{4}.$$

when $a = 2$, $k = -3$

$$g(x) = x^2 - (-3+1)x + \frac{4+9-6}{4}$$

$$= x^2 + 2x + \frac{7}{4} \quad 5$$

$$y = 2 - g(x)$$

$$= 2 - x^2 - 2x - \frac{7}{4}$$

$$= -x^2 - 2x + \frac{1}{4}$$

$$= -[x^2 + 2x - \frac{1}{4}]$$

$$= -[(x+1)^2 - \frac{1}{4} - 1]$$

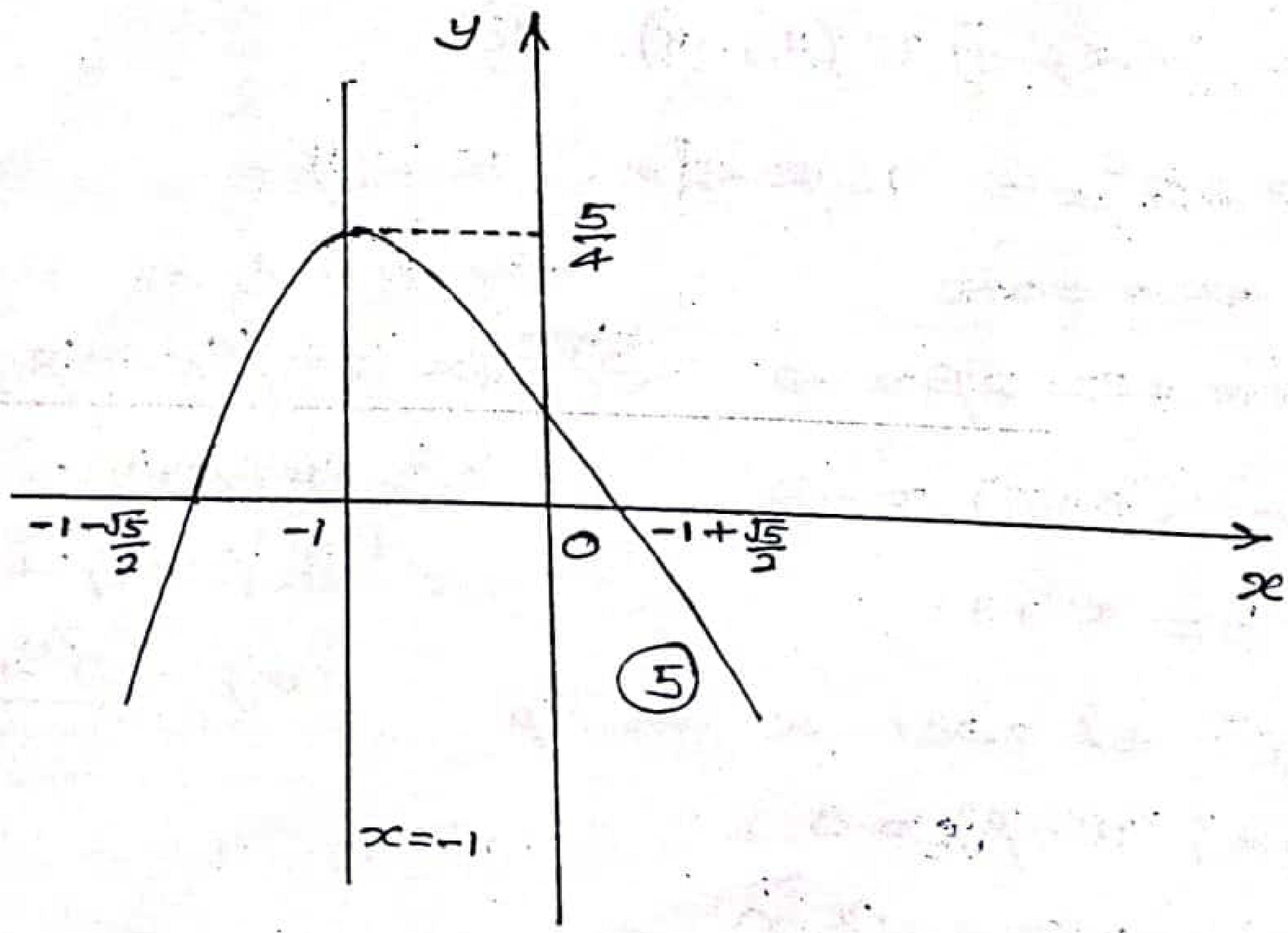
$$= \frac{5}{4} - (x+1)^2 \quad 5$$

For maximum value of y, $(x+1)^2 = 0$

$$y_{max} = \frac{5}{4} \quad 5$$

the eq^n of symmetric axis.

$x = -1$ 5



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(b) $f(x) = x^4 + 6x^3 + \lambda x^2 + \mu x + \nu$

since $(x-2)$ and $(x+1)$ are factors of $f(x)$

$f(2) = 0$ (5) $\quad f(-1) = 0$ (5)

$16 + 48 + 4\lambda + 2\mu + \nu = 0$

$4\lambda + 2\mu + \nu = -64$ — ① (5)

$1 - 6 + \lambda - \mu + \nu = 0$

$\lambda - \mu + \nu = 5$ — ② (5)

the remainder is 10 when $f(x)$ divided by $(x-1)$

$f(1) = 10$ (5)

$1 + 6 + \lambda + \mu + \nu = 10$

$\lambda + \mu + \nu = 3$ — ③ (5)

② − ③

$-2\mu = 2$

$\mu = -1$ (5)

① − ②

$3\lambda + 3\mu = -69$

$3\lambda - 3 = -69$ (5)

$\lambda = -22$

③ $\Rightarrow$ $-22 - 1 + \nu = 3$

$\nu = 26$ (5)

$f(x) = x^4 + 6x^3 - 22x^2 - x + 26$ (5) $\boxed{50}$

$f(x) = g(x)(x^2 - 5x + 6) + Ax + B$

$= g(x)(x-2)(x-3) + Ax + B$

when $x = 2$

$f(2) = 2A + B$ (5)

$16 + 48 - 88 - 2 + 26 = 2A + B$

$2A + B = 0$ — (*)

when $x = 3$

$f(3) = 3A + B$ (5)

$81 + 162 - 198 - 3 + 26 = 3A + B$

$3A + B = 68$ — (**)

(*), (**)

$A = 68$

$B = -136$

Remainder

$68x - 136$

(5) $\boxed{15}$

12 (a) A, B, C 1, 2, 3

(i) The number of arrangements using all letters and all digits given abou

$= 6!$ 5

$= 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 720$ 5

10

(ii) The number of arrangements that using two letters and two digits at a time 5 5

$^3C_2 \times {^3C_2} \times 4!$ $^3P_2 \times {^3P_2} = \frac{3!}{1!} \times \frac{3!}{\cancel{2}!} = 36$ 10

216

(iii) The number of arrangements that start from letter and one after one there should be a d

$^3C_1 \times {^3C_1} \times {^2C_1} \times {^2C_1} \times {^1C_1} \times {^1C_1}$ 10

$= 3 \times 3 \times 2 \times 2 \times 1 \times 1 = 36$ 5

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(b) $\frac{3}{1.2.4} + \frac{4}{2.3.5} + \frac{5}{3.4.6} + \cdots$

$u_r = \frac{r+2}{r(r+1)(r+3)}$ 10

$V_r = \frac{Ar^2 + Br + C}{(r+1)(r+2)(r+3)}$

$V_{r-1} = \frac{A(r-1)^2 + B(r-1) + C}{r(r+1)(r+2)}$ 5

10

$V_r - V_{r-1} = \frac{Ar^2 + Br + C}{(r+1)(r+2)(r+3)} - \frac{A(r^2-2r+1) + B(r-1) -}{r(r+1)(r+2)}$

$= \frac{1}{(r+1)(r+2)} \left[\frac{r(Ar^2+Br+C) - (r+3)[A(r^2-2r+1)+B(}{r(r+3)} \right.$

$= \frac{-Ar^2 + 5Ar - 2Br - 3A + 3B - 3C}{r(r+1)(r+2)(r+3)}$ 5

$$\frac{r+2}{r(r+1)(r+3)} = \frac{-Ar^2+(5A-2B)r-3A+3B-3C}{r(r+1)(r+2)(r+3)} \quad 5$$

$$r^2+4r+4 = -Ar^2+(5A-2B)r-3A+3B-3C \quad 5$$

By comparing coefficients

r^2 : $-A = 1 \Rightarrow A = -1$ 5

r^1 : $5A-2B = 4$ 10 $\Rightarrow -5-2B = 4$

$B = -\frac{9}{2}$ 5

r^0 : $-3A+3B-3C = 4$

$\Rightarrow 3 - \frac{27}{2} - 3C = 4$

$-3C = 4 + \frac{21}{2}$

$C = -\frac{29}{6}$ 5

$\boxed{65}$

$u_r = v_r - v_{r-1}$

$V_0 = \frac{-\frac{29}{6}}{1\cdot 2\cdot 3}$

$= -\frac{29}{36}$ 5

$r=1$ $u_1 = v_1 - v_0$ 5

$r=2$ $u_2 = v_2 - v_1$

$r=n-1$ $u_{n-1} = V_{n-1} - V_{n-2}$ 5

$r=n$ $u_n = V_n - V_{n-1}$

$V_n = \frac{-r^2 - \frac{9}{2}r - \frac{29}{6}}{(n+1)(n+2)(n+3)}$

$= -\frac{6n^2+27n+29}{6(n+1)(n+2)(n+3)}$ 5

$\sum_{r=1}^{n} u_r = v_n - v_0$ 5

$\sum_{r=1}^{n} u_r = \frac{29}{36} - \frac{6n^2+27n+29}{6(n+1)(n+2)(n+3)}$ 5

$$\lim_{n\to\infty} \sum_{r=1}^{n} u_r = \frac{29}{36} - \underbrace{\lim_{n\to\infty} \frac{n^2\left(6+\frac{27}{n}+\frac{29}{n^2}\right)}{6n\left(1+\frac{1}{n}\right)n\left(1+\frac{2}{n}\right)n\left(1+\frac{3}{n}\right)}}_{0} \quad 5$$

$\sum_{r=1}^{\infty} u_r = \frac{29}{36}$ 5

The infinite series is convergent. 5

The sum is $\frac{29}{36}$ 5

$\boxed{50}$

13. (a)

$$A = \begin{pmatrix} 1 & 1 \\ p & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & q \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & p & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad (5)$$

$$A^T A = B$$

$$\begin{pmatrix} 1 & p & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & q \end{pmatrix} \quad (5)$$

$$\begin{pmatrix} 1+p^2 & 1 \\ 1 & 2 \end{pmatrix} \; (5) = \begin{pmatrix} 2 & 1 \\ 1 & q \end{pmatrix}$$

By comparing elements

$$1 + p^2 = 2$$

$$p^2 = 1$$

$$p = 1 \quad (5) \qquad q = 2 \quad (5)$$

2

$$X = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} \qquad X^{-1} = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix} \quad (5)$$

$$XC = BX$$

$$\underbrace{X^{-1}X}_{I}C = X^{-1}BX$$

(5)

$$C = X^{-1}BX$$

$$= \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} \quad (5)$$

$$= \begin{pmatrix} 5 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} \quad (5)$$

$$= \begin{pmatrix} 3 & 7 \\ 0 & 1 \end{pmatrix} \quad (5)$$

2

(b) $z = -1 + \sqrt{3}\, i$

$\bar{z} = -1 - \sqrt{3}\, i$ 5 $\qquad |z| = \sqrt{(-1)^2 + \sqrt{3}^2}$

$= 2$ 5 $\qquad \boxed{10}$

$z\bar{z} = (-1 + \sqrt{3}\, i)(-1 - \sqrt{3}\, i)$ 5

$= 1 - 3i^2$

$= 4$ 5 $\qquad \boxed{10}$

$= |z|^2$

$z = 2\left[-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right]$ 5

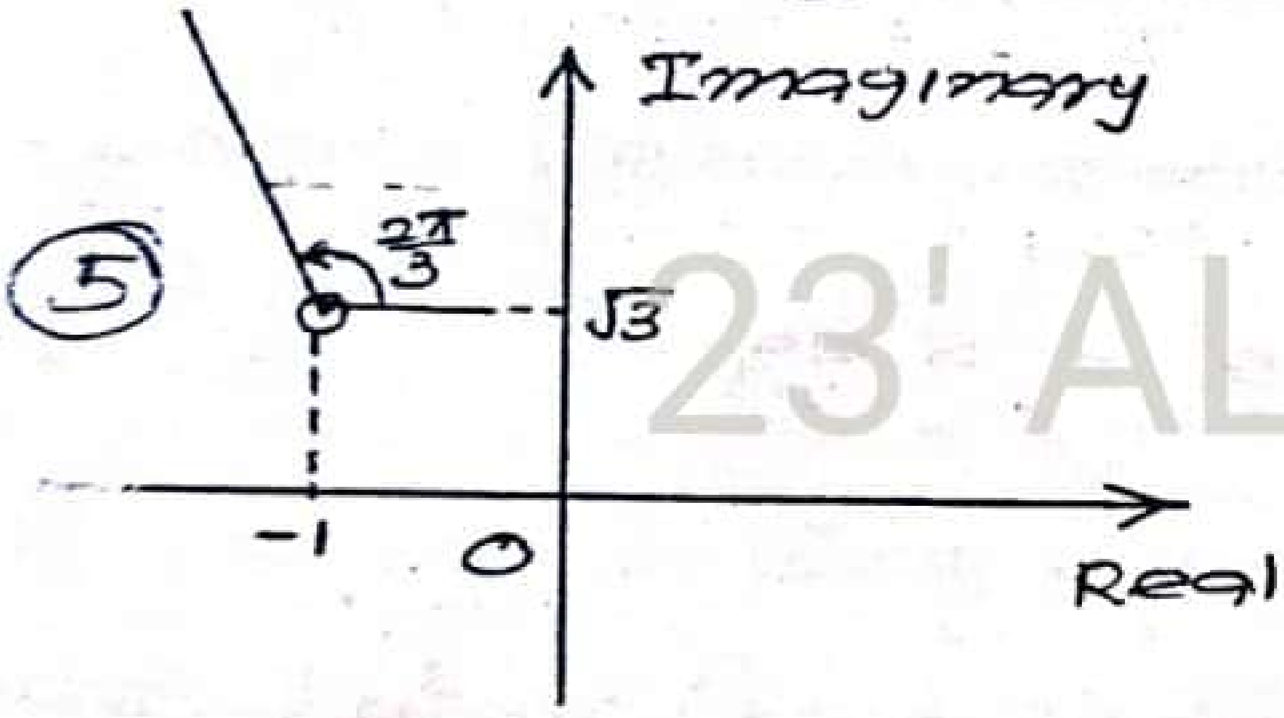
$= 2\left[-\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right]$ 5

$= 2\left[\cos\left(\pi - \frac{\pi}{3}\right) + i \sin\left(\pi - \frac{\pi}{3}\right)\right]$ 5

$= 2\left[\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}\right]$

$\operatorname{Arg}(z) = \frac{2\pi}{3}$ 5

5



$\boxed{25}$

$z_1 = 2\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)$

$= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$

$= 1 + \sqrt{3}\, i$ 5

$z + z_1 = -1 + \sqrt{3}\, i + 1 + \sqrt{3}\, i$

$= 2\sqrt{3}\, i$ 5

$z + z_1$ is purely imaginary. 5 $\qquad \boxed{15}$

(c) De Moivre's Theorem

Let $z = \cos\theta + i\sin\theta$. For any positi[ve] integer n, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ 5

$$z^5 = (\cos\theta + i\sin\theta)^5$$

$$= \cos 5\theta + i\sin 5\theta \quad 5$$

$$z^5 = {}^5C_0\cos^5\theta + {}^5C_1\cos^4\theta\, i\sin\theta + {}^5C_2\cos^3\theta\, i^2\sin^2\theta$$

$$+ {}^5C_3\cos^2\theta\, i^3\sin^3\theta + {}^5C_4\cos\theta\, i^4\sin^4\theta$$

$$+ {}^5C_5\, i^5\sin^5\theta \quad 5$$

$$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) \quad 5$$

$$+ i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$$

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

$$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$$

$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta \quad 5$$

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$

$$= 5\sin\theta - 10\sin^3\theta - 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$

$$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta \quad 5$$

$$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta} = \frac{16\sin^5\theta - 20\sin^3\theta + 5\sin\theta}{16\cos^5\theta - 20\cos^3\theta + 5\cos\theta} \quad 5$$

$$= \frac{\tan\theta(16\sin^4\theta - 20\sin^2\theta + 5)}{16\cos^4\theta - 20\cos^2\theta + 5}$$

$$\tan 5\theta = \frac{\tan\theta\,(16\tan^4\theta - 20\tan^2\theta(1+\tan^2\theta) + 5(1+\tan^2\theta)^2}{16 - 20(1+\tan^2\theta) + 5(1+\tan^2\theta)^2}$$

$$= \frac{\tan\theta\,[\tan^4\theta - 10\tan^2\theta + 5]}{5\tan^4\theta - 10\tan^2\theta + 1}$$ $\boxed{35}$

4.(a)

$$f(x) = \frac{12x^2 - 3x + 1}{4(x-1)^3} \qquad x \neq 1$$

differentiate w.r.t x,

$$f'(x) = \frac{4(x-1)^3(24x-3) - (12x^2-3x+1)\,12(x-1)^2}{16(x-1)^6}$$ 15

$$= \frac{3(x-1)(8x-1) - 3(12x^2-3x+1)}{4(x-1)^4}$$

$$= \frac{3}{4} \cdot \frac{8x^2 - 9x + 1 - (12x^2-3x+1)}{(x-1)^4}$$

$$= \frac{3}{4} \cdot \frac{-4x^2 - 6x}{(x-1)^4}$$

$$= \frac{3}{4} \cdot \frac{-2x(2x+3)}{(x-1)^4}$$

$$= \frac{-3x(2x+3)}{2(x-1)^4}$$ 5 $\boxed{20}$

For turning points $f'(x) = 0$ 5

$$\frac{-3x(2x+3)}{2(x-1)^4} = 0$$

$\Rightarrow \quad x = 0$ 5 , $x = -\frac{3}{2}$ 5

Let $y = \dfrac{12x^2 - 3x + 1}{4(x-1)^3}$

when $x = 0$

$y = -\frac{1}{4}$

$(0, -\frac{1}{4})$ 5

when $x = -\frac{3}{2}$

$$y = \frac{12 \cdot \frac{9}{4} - 3(-\frac{3}{2}) + 1}{4(-\frac{3}{2} - 1)^3}$$

$$= -\frac{13}{25}$$

$(-\frac{3}{2}, -\frac{13}{25})$ 5

$\boxed{25}$

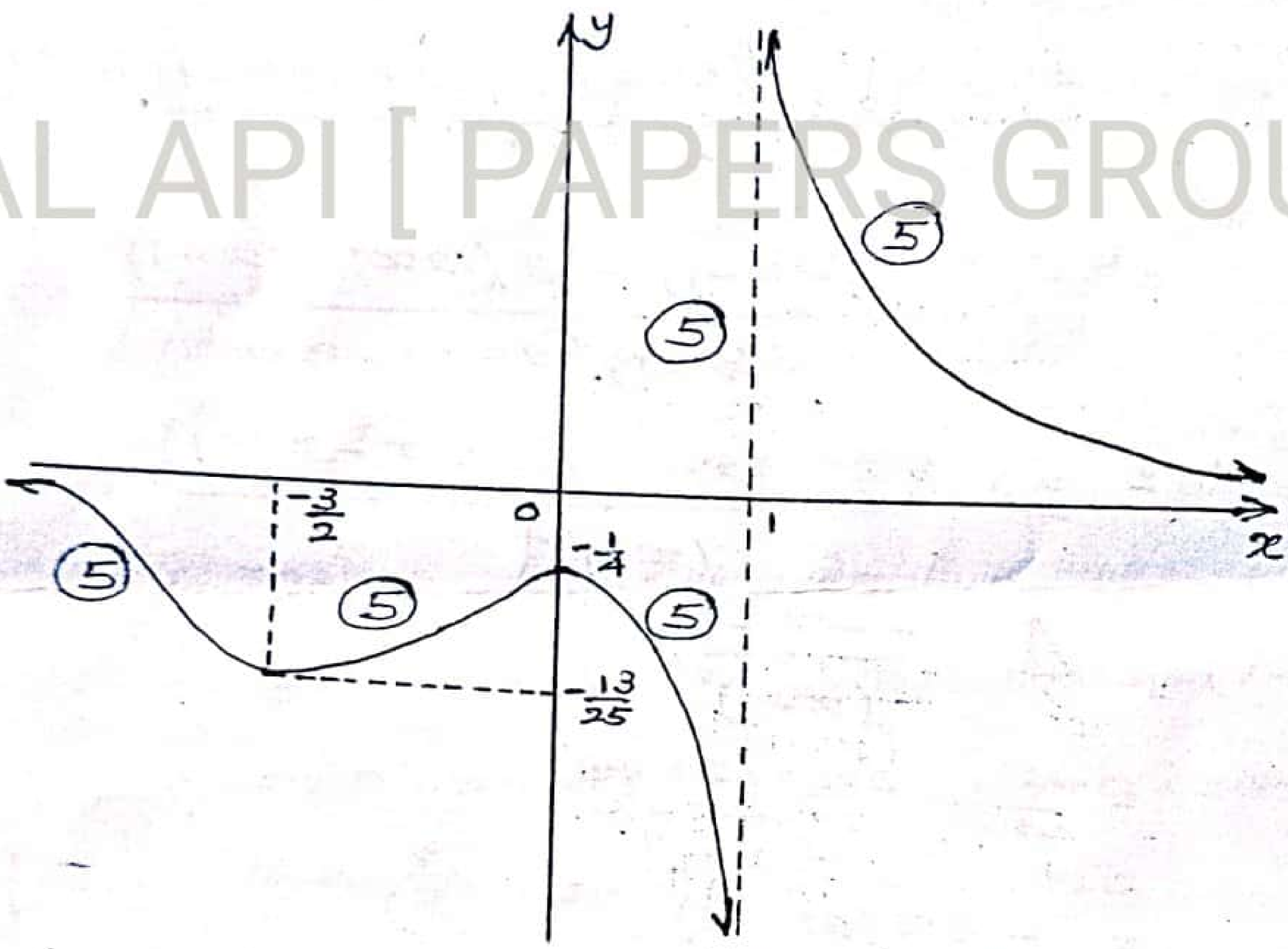
$x = -\frac{3}{2}$ $\qquad\qquad$ $x = 0$

	$x < \frac{3}{2}$	$-\frac{3}{2} < x < 0$	$0 < x < 1$	$x > 1$
sign of $f'(x)$	$(-)$ $x\uparrow\ f(x)\downarrow$ $\searrow$	$(+)$ $x\uparrow\ f(x)\uparrow$ $\nearrow$	$(-)$ $x\uparrow\ f(x)\downarrow$ $\searrow$	$(-)$ $x\uparrow\ f(x)\downarrow$ $\searrow$

5 $\qquad$ 5 $\qquad$ 5

At $x = -\frac{3}{2}$ is a minimum 5 $\qquad$ At $x = 0$ is a maximum 5

2



$$y = \frac{12x^2 - 3x + 1}{4(x-1)^3}$$

$$= \frac{x^2\left[12 - \frac{3}{x} + \frac{1}{x^2}\right]}{4x^3\left(1 - \frac{1}{x}\right)^3}$$

$x \to +\infty$

$$y = \frac{12 - \cancel{\frac{3}{\infty}}_0 + \cancel{\frac{1}{\infty^2}}_0}{4\infty\left(1 - \cancel{\frac{1}{\infty}}_0\right)^3}$$

$y \to 0$

5

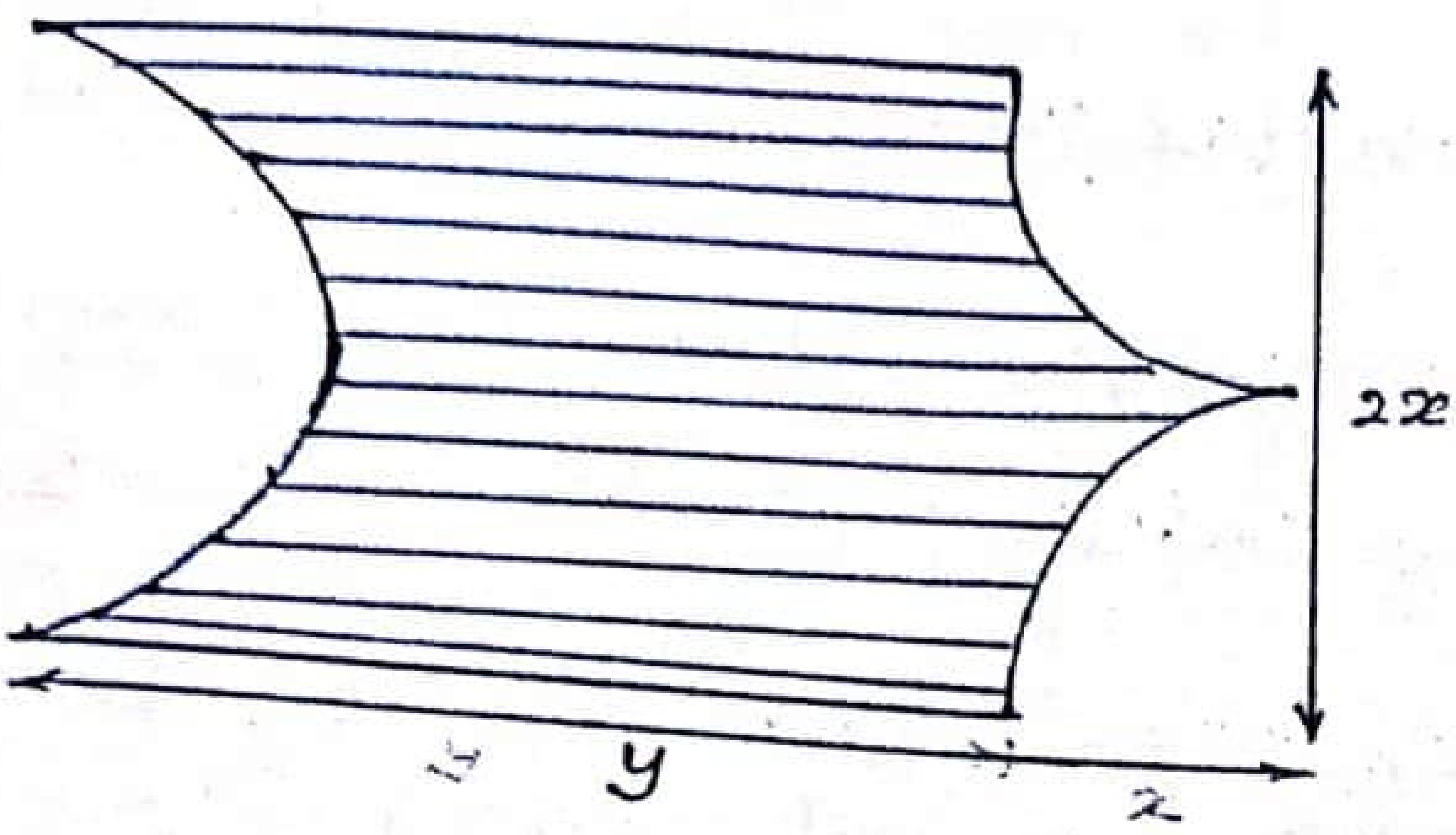
$x \to -\infty$

$$y = \frac{12 - \cancel{\frac{3}{-\infty}}_0 + \cancel{\frac{1}{(-\infty)^2}}_0}{4(-\infty)\left(1 - \cancel{\frac{1}{\infty}}_0\right)^3}$$

$y \to 0$

30

(b)



shaded area $\quad A = 2xy - \pi x^2$ 5

perimeter $\quad P = 2\pi x + 2(y-x)$ 5

$$y = \frac{p+2x-2\pi x}{2} \quad 5$$

$$A = 2x \cdot \frac{p+2x-2\pi x}{2} - \pi x^2 \quad 5$$

$$= px + 2x^2 - 3\pi x^2$$

$$\frac{dA}{dx} = p + 4x - 6\pi x \quad 5$$

For max$^{\underline{m}}$ / min$^{\underline{m}}$ $\quad \frac{dA}{dx} = 0$ 5

$$p + 4x - 6\pi x = 0$$

$$x = \frac{p}{2(3\pi-2)} \quad 5$$

	$0 < x < \frac{p}{2(3\pi-2)}$	$x = \frac{p}{2(3\pi-2)}$	$x > \frac{p}{2(3\pi-2)}$
sign of $\frac{dA}{dx}$	(+) $x\uparrow$ $A\uparrow$ /	0 —	(−) $x\uparrow$ $A\downarrow$ \

5 5

When $x = \frac{p}{2(3\pi-2)}$, the shaded area is maximum. 5

$\boxed{50}$

15. (a)

$$3x^3+x^2+7x+3 = A(x+1)(x^2+5) + B(x^2+5) + (x+1)^2(Cx-2)$$ (5)

$$= A(x^3+x^2+5x+5) + B(x^2+5) + (x^2+2x+1)(Cx-2)$$

$$= A(x^3+x^2+5x+5) + B(x^2+5) + Cx^3-2x^2+2Cx^2-4x+C-2$$

$$= (A+C)x^3 + (A+B+2C-2)x^2 + (5A+C-4)x + 5A+5B-2$$ (5)

By comparing coefficients

x^3: $A+C=3$ — (1)

(5) x^2: $A+B+2C-2=1 \Rightarrow A+B+2C=3$ — (2)

x^1: $5A+C-4=7 \Rightarrow 5A+C=11$ — (3)

x^0: $5A+5B-2=3 \Rightarrow A+B=1$ — (4)

(3) − (1): $4A=8$, $A=2$ (5)

(4) ⇒ $2+B=1$, $B=-1$ (5)

(1) ⇒ $2+C=3$, $C=1$ (5)

$\boxed{30}$

$$3x^3+x^2+7x+3 = 2(x+1)(x^2+5) - (x^2+5) + (x+1)^2(x-2)$$

$\div (x+1)^2(x^2+5)$

$$\frac{3x^3+x^2+7x+3}{(x+1)^2(x^2+5)} = \frac{2}{x+1} - \frac{1}{(x+1)^2} + \frac{x-2}{x^2+5}$$ (5)

$$\int \frac{3x^3+x^2+7x+3}{(x+1)^2(x^2+5)}\,dx = 2\int\frac{dx}{x+1} - \int\frac{dx}{(x+1)^2} + \int\frac{x-2}{x^2+5}\,dx$$

$$= 2\ln|x+1| + \frac{1}{x+1} + \frac{1}{2}\int\frac{2x}{x^2+5}\,dx - 2\int\frac{dx}{x^2+5}$$

$$= 2\ln|x+1| + \frac{1}{x+1} + \frac{1}{2}\ln|x^2+5| - \frac{2}{\sqrt5}\tan^{-1}\frac{x}{\sqrt5} + c$$

(5) (5) (5) (5)

where c is the arbitrary constant (5)

$\boxed{30}$

$$I=\int \cos 2x \ \ln|1+\tan x|\,dx$$

$$= \int \ln|1+\tan x| \ \frac{d}{dx}\left(\frac{\sin 2x}{2}\right) dx$$

5

$$= \ln|1+\tan x| \ \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \cdot \frac{1}{1+\tan x} \sec^2 x \, dx$$

5

$$= \frac{\sin 2x \ \ln|1+\tan x|}{2} - \int \frac{\sin x \cos x}{\cos^2 x} \cdot \frac{1}{1+\tan x}\,dx$$

$$= \frac{\sin 2x \ \ln|1+\tan x|}{2} - \underbrace{\int \frac{\tan x}{1+\tan x}\,dx}_{I_1} \quad \text{—} (*)$$

5

$$I_1 = \int \frac{\tan x}{1+\tan x}\,dx$$

substitution

$$t = \tan x$$

$$dt = \sec^2 x\,dx$$

$$dx = \frac{dt}{1+t^2}$$

$$= \int \frac{t}{1+t} \cdot \frac{dt}{1+t^2}$$

5

$$\frac{t}{(t+1)(t^2+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+1}$$

$$t = A(t^2+1) + (t+1)(Bt+C)$$

$$t = (A+B)t^2 + (B+C)t + A + C$$

By comparing coefficients

t^2 : $A+B=0$ $\qquad A = -\frac{1}{2}$

t^1 : $B+C=1$ $\qquad B = \frac{1}{2}$ $\qquad$ 10

t^0 : $A+C=0$ $\qquad C = \frac{1}{2}$

$$I_1 = -\frac{1}{2}\int \frac{dt}{t+1} + \frac{1}{2}\int \frac{t+1}{t^2+1}\,dt$$

$$= -\frac{1}{2}\ln|t+1| + \frac{1}{2}\left[\frac{1}{2}\int \frac{2t}{t^2+1}\,dt + \int \frac{dt}{t^2+1}\right]$$

$$= -\frac{1}{2}\ln|t+1| + \frac{1}{4}\ln|t^2+1| + \frac{1}{2}\tan^{-1} t$$

5 $\quad$ 5 $\quad$ 5

$(*) \Rightarrow$

$$I = \frac{\sin 2x \ln|1+\tan x|}{2} + \frac{1}{2}\ln|1+\tan x| - \frac{1}{4}\ln|\tan^2 x+1| - \frac{1}{2}\tan^{-1}(\tan x) + c,$$

5

$\boxed{50}$

(c) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$RHS = \int_0^a f(a-x)\,dx$ Let $a-x=y$

$-dx = dy$

$= \int_a^0 f(y)\,(-)\,dy$ 5

$= \int_0^a f(y)\,dy$

Replace $y \rightarrow x$

$= \int_0^a f(x)\,dx = LHS.$ 5

10

$$I = \int_0^\pi \frac{x\cos 2x}{1+\cos 2x}\,dx$$

$$= \int_0^\pi \frac{(\pi-x)\cos 2(\pi-x)}{1+\cos 2(\pi-x)}\,dx \quad 5$$

$$= \pi\int_0^\pi \frac{\cos 2x}{1+\cos 2x}\,dx - \underbrace{\int_0^\pi \frac{x\cos 2x}{1+\cos 2x}\,dx}_{I} \quad 5$$

$$2I = \pi\int_0^\pi \frac{\cos 2x}{1+\cos 2x}\,dx$$

$$I = \frac{\pi}{2}\int_0^\pi \frac{1-2\sin^2 x}{1+2\cos^2 x-1}\,dx \quad 5$$

$$= \frac{\pi}{4}\int_0^\pi (\sec^2 x - 2\tan^2 x)\,dx$$

$$= \frac{\pi}{4}\int_0^\pi \sec^2 x - 2(\sec^2 x - 1)\,dx$$

$$= \frac{\pi}{4}\int_0^\pi (2-\sec^2 x)\,dx \quad 5$$

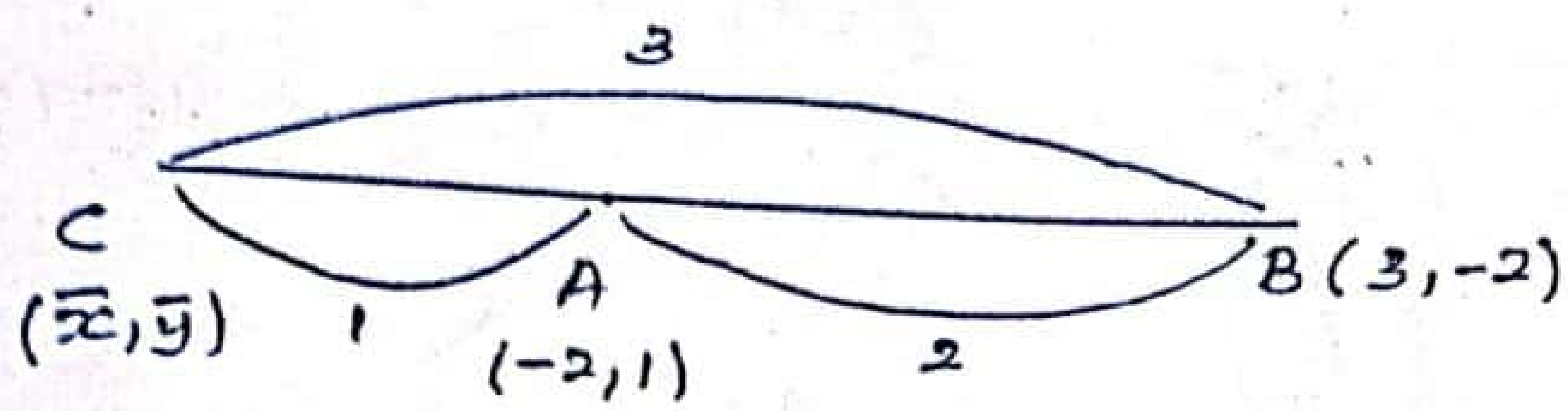
$$= \frac{\pi}{4}\left[2x - \tan x\right]_0^\pi \quad 5$$

$$= \frac{\pi}{4}\left[(2\pi - \tan\pi) - 0\right]$$

$$= \frac{\pi^2}{2} \quad 5$$

30

16.

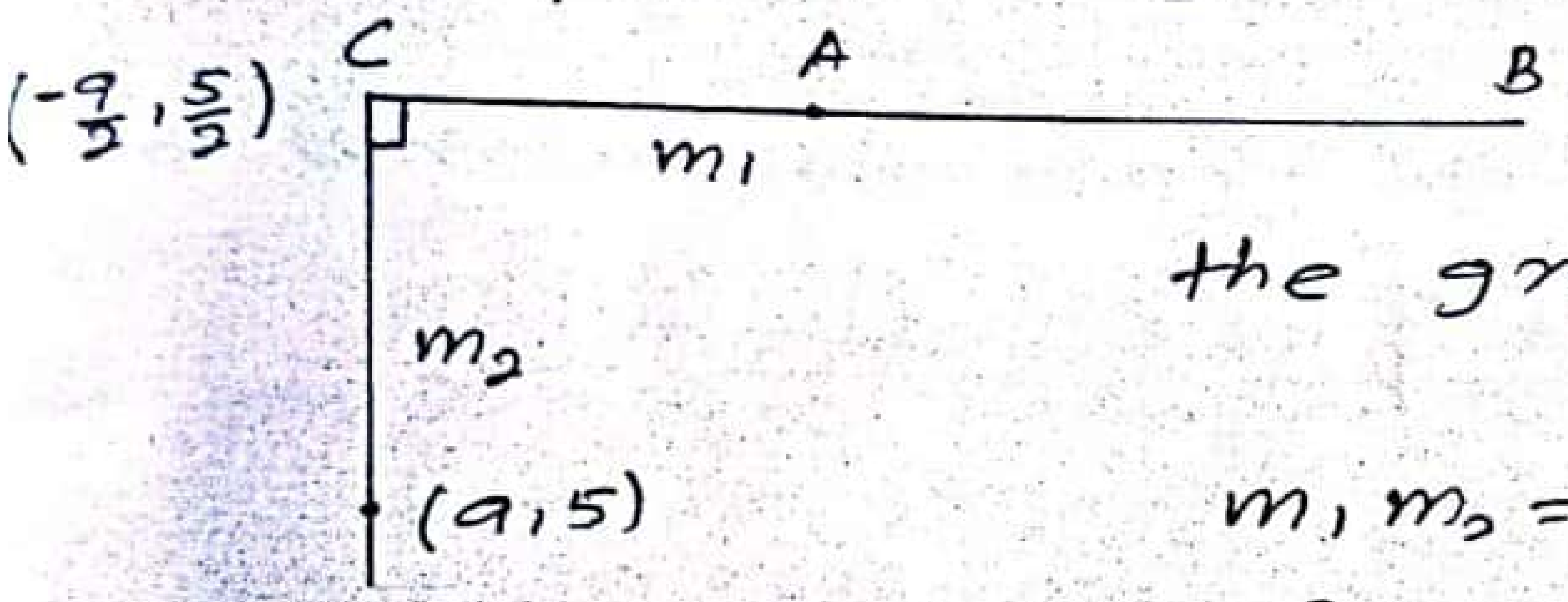


$$\frac{3+2\bar{x}}{3} = -2 \qquad\qquad \frac{-2+2\bar{y}}{3} = 1$$

$$\bar{x} = -\frac{9}{2} \quad (5) \qquad\qquad \bar{y} = \frac{5}{2} \quad (5)$$

$$C \equiv \left(-\frac{9}{2}, \frac{5}{2}\right) \quad (5)$$

$\boxed{15}$



the gradient of AB, $m_1 = \frac{-2-1}{3+2}$ (5)

$$= -\frac{3}{5}$$

$$m_1 m_2 = -1 \quad (5)$$

$$-\frac{3}{5} m_2 = -1$$

$$m_2 = \frac{5}{3} \quad (5)$$

the eq^n of the line $\perp^r$ to AB

$$\frac{y-\frac{5}{2}}{x+\frac{9}{2}} = \frac{5}{3} \quad (5)$$

$$\frac{2y-5}{2x+9} = \frac{5}{3}$$

$$10x+45 = 6y-15$$

$$10x-6y+60=0$$

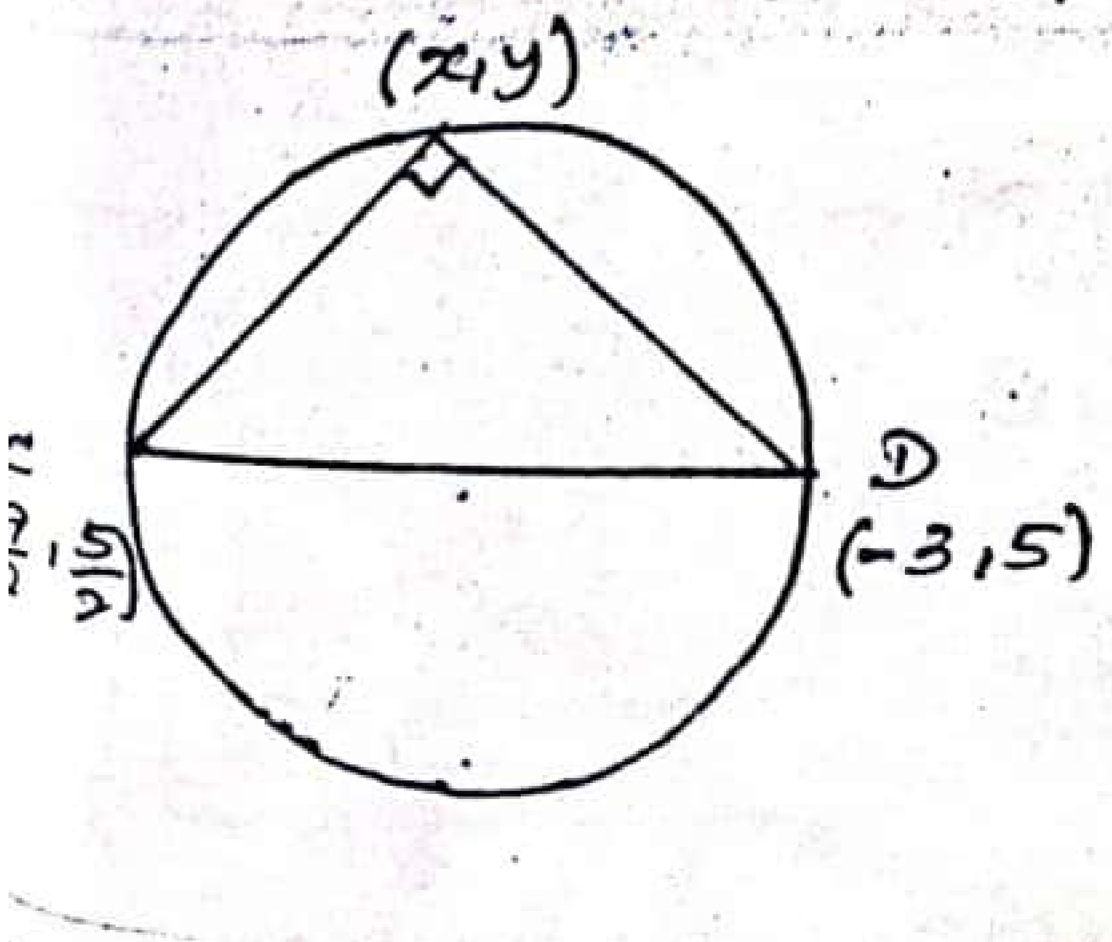
$$5x-3y+30=0 \quad (5)$$

(a,5) satisfy the eq^n

$$5a-15+30=0$$

$$a=-3 \quad (5)$$

$\boxed{30}$



the eq^n of the circle

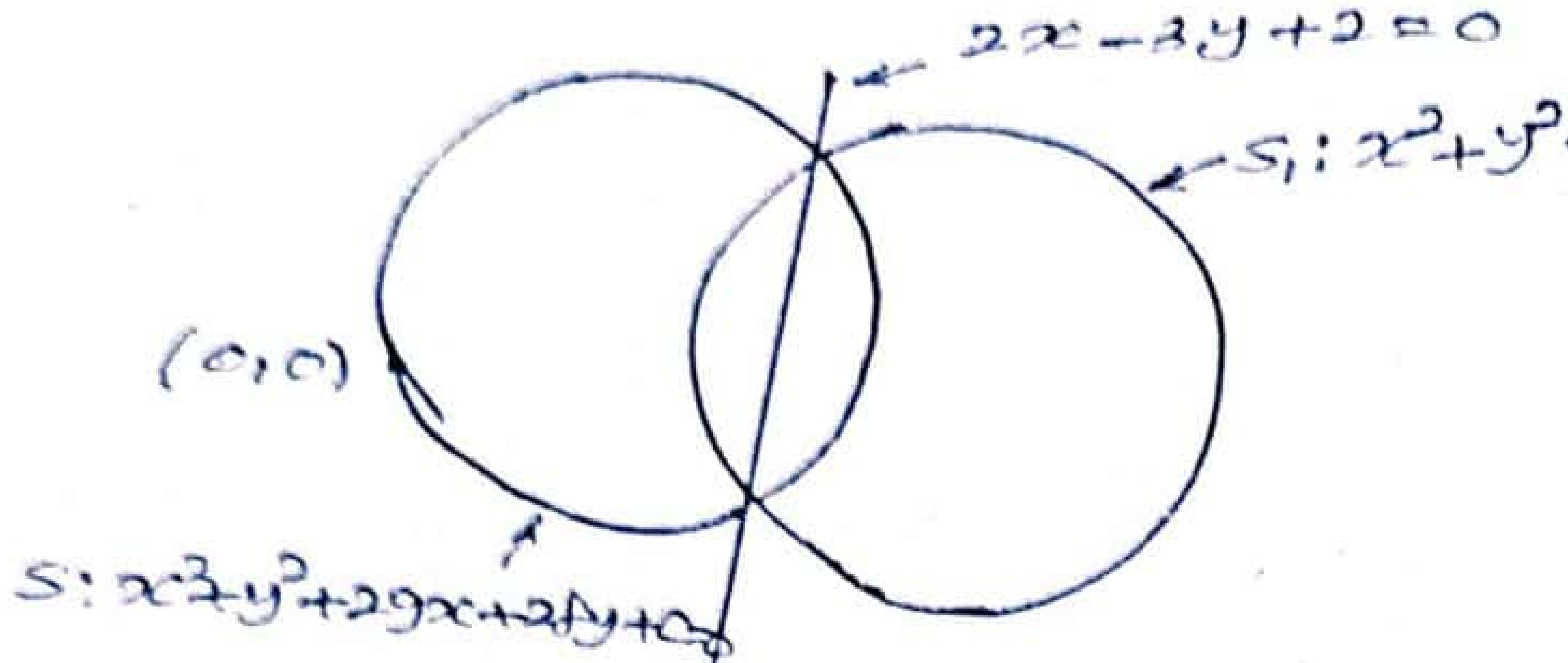
$$(5)\ \left(\frac{y-\frac{5}{2}}{x+\frac{9}{2}}\right)\left(\frac{y-5}{x+3}\right) (5) = -1 \quad (5)$$

$$2y^2-15y+25 = -(2x^2+15x+27)$$

$$2x^2+2y^2+15x-15y+52=0$$

(5)

$\boxed{20}$



(0,0), satisfy the eq^n $S=0$

$0+0+0+0+c=0 \Rightarrow c=0$ (10)

For the eq^n of the common chord $S-S_1$

$(2g-\frac{15}{2})x+(2f+\frac{15}{2})y+c-26=0$ (10)

$2x-3y+2=0$

$\frac{2g-\frac{15}{2}}{2}=\frac{2f+\frac{15}{2}}{-3}=\frac{c-26}{2}=\lambda$ (15)

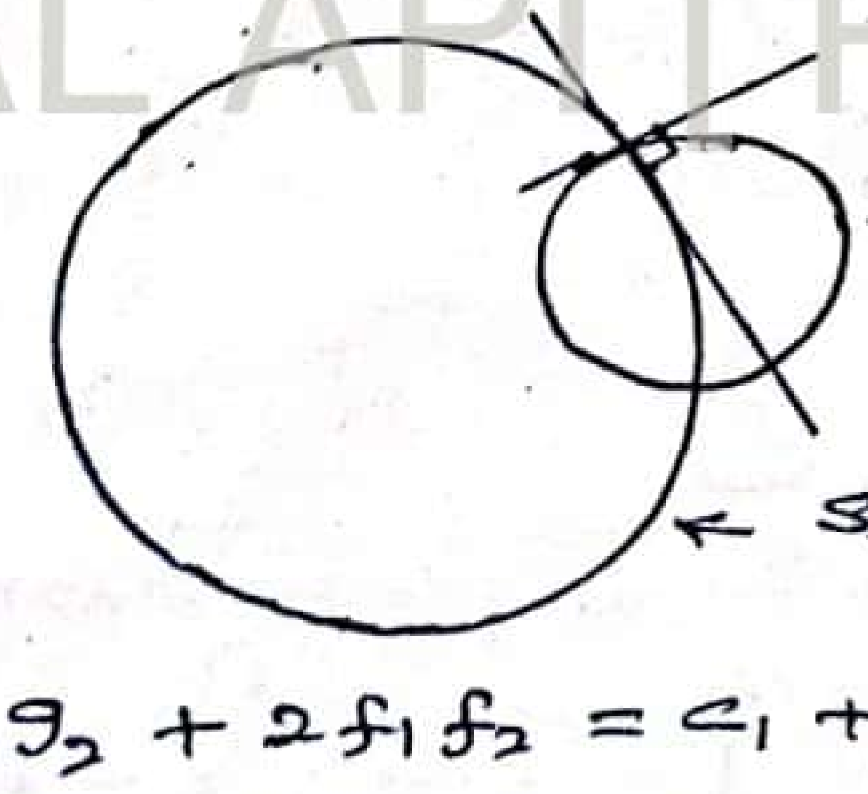
$\Rightarrow \lambda=-13$ (5)

$2g-\frac{15}{2}=-26 \Rightarrow g=-\frac{37}{4}$ (5)

$2f+\frac{15}{2}=39 \Rightarrow f=\frac{63}{4}$ (5)

$x^2+y^2+2(-\frac{37}{4})x+2\cdot(\frac{63}{4})y+0=0$

$S: x^2+y^2+\frac{37}{2}x+\frac{63}{2}y=0$ (5) $\boxed{55}$



$2g_1g_2+2f_1f_2=c_1+c_2$ (10)

$2(-)\frac{37}{4}\cdot\frac{k}{2}+2\cdot\frac{63}{4}\cdot 1=0-1$ (10)

$-37k+126=-4$

$k=\frac{130}{37}$ (10) $\boxed{30}$

7.(a)

$\sin(A+B) = \sin A \cos B + \cos A \sin B$ 5

$\sin(90-\theta) = \cos\theta$

Let $\theta = A+B$

$\sin[90-(A+B)] = \cos(A+B)$ — (1) 5

5

$\sin[(90-A)+(-B)] = \sin(90-A)\cos(-B) + \cos(90-A)\sin(-B)$

5

$= \cos A \cos B + \sin A (-) \sin B$

5

$= \cos A \cos B - \sin A \sin B$ — (2)

(1) & (2) $\Rightarrow$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$ 5 $\boxed{30}$

$\cos 105$

$= \cos(60+45)$ 5

$= \cos 60 \cos 45 - \sin 60 \sin 45$

$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}$

$= \frac{1-\sqrt{3}}{2\sqrt{2}}$ 5

$\sin 75$

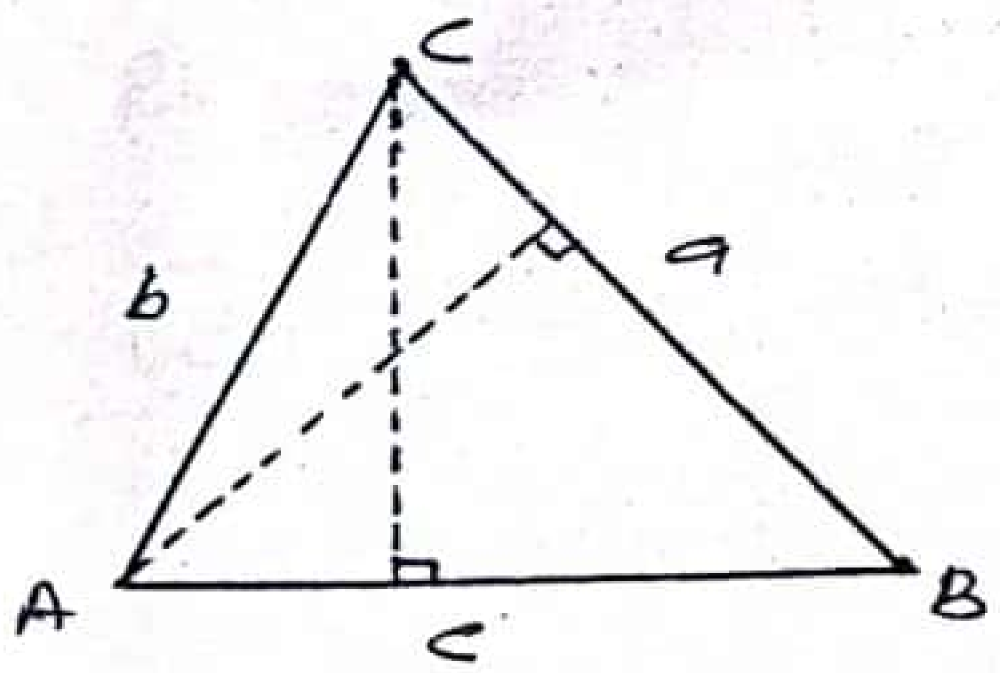
$= \sin(45+30)$ 5

$= \sin 45 \cos 30 + \cos 45 \sin 30$

$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$ 5 $\boxed{20}$

For acute angle triangle



$b \sin A = a \sin B$

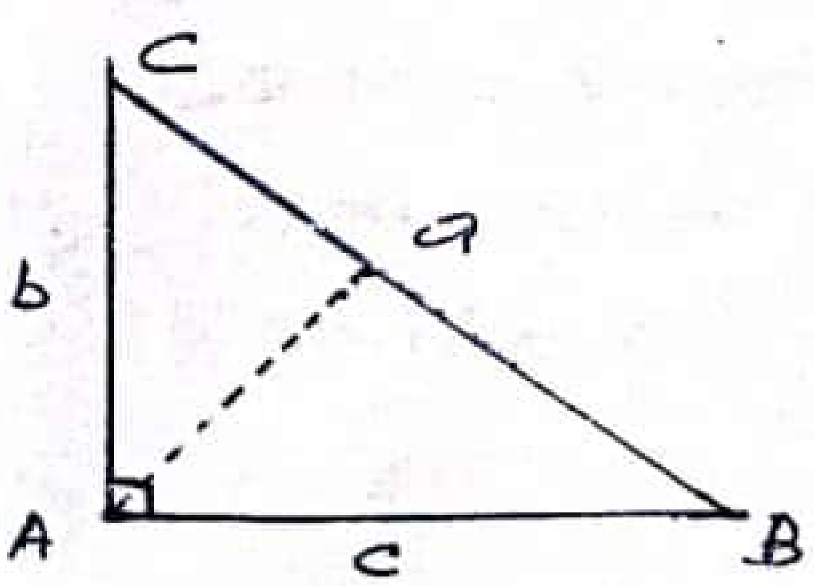
$\frac{\sin A}{a} = \frac{\sin B}{b}$ 5

$b \sin C = c \sin B$

$\frac{\sin C}{c} = \frac{\sin B}{b}$

$\Rightarrow \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

For right angle triangle



$b \sin C = c \sin B$

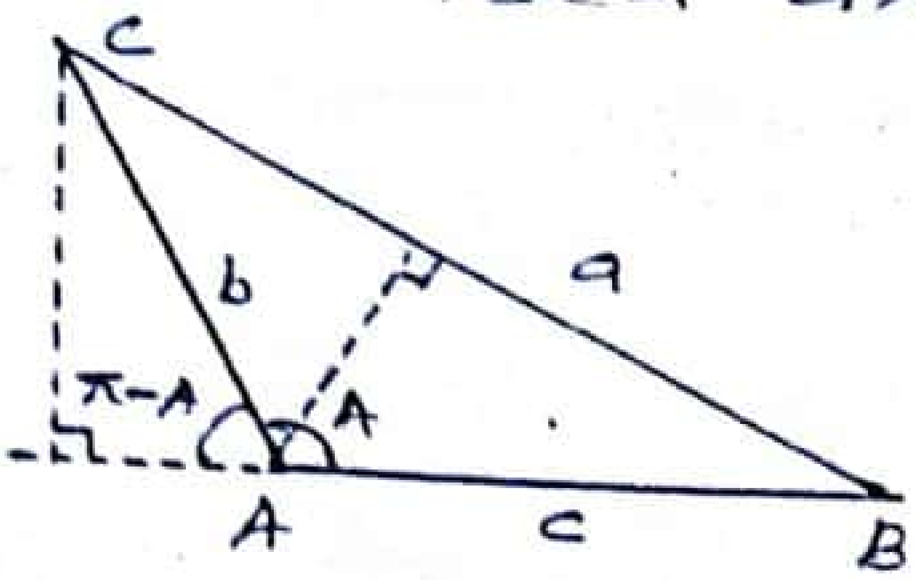
$\frac{\sin C}{c} = \frac{\sin B}{b}$ 5

$a \sin B = b \sin 90$

23 $\frac{\sin B}{b} = \frac{\sin A}{a}$

$\Rightarrow \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

For obtused angle triangle



$b \sin C = c \sin B$

$\dfrac{\sin C}{c} = \dfrac{\sin B}{b}$ 5

$a \sin B = b \sin(\pi - A) = b \sin A$

$\dfrac{\sin B}{b} = \dfrac{\sin A}{a}$

$\Rightarrow \dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

(b) $c = (a-b)\cos\frac{C}{2}\,\text{cosec}\left(\frac{A-B}{2}\right)$ 5

$\boxed{20}$

$\dfrac{c}{a-b} = \cos\frac{C}{2}\,\text{cosec}\,\frac{A-B}{2}$

$LHS = \dfrac{c}{a-b}$ 5

$= \dfrac{\frac{\sin C}{k}}{\frac{\sin A}{k} - \frac{\sin B}{k}}$ 5 $= \dfrac{\sin C}{2\cos\left(\frac{A+B}{2}\right)\sin\frac{A-B}{2}}$ 5

$= \dfrac{2\sin\frac{C}{2}\cos\frac{C}{2}}{2\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\sin\left(\frac{A-B}{2}\right)}$ 5 5

$= \cos\frac{C}{2}\,\text{cosec}\left(\frac{A-B}{2}\right) = RHS$

(c) $\tan^{-1}(\sin^2 2x) + \tan^{-1}(\cos 2x) = \frac{\pi}{4}$ 5

$\boxed{30}$

$\alpha + \beta = \frac{\pi}{4}$

$\tan(\alpha+\beta) = \tan\frac{\pi}{4}$ 5

$\dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = 1$ 5

$\sin^2 2x + \cos 2x = 1 - \sin^2 2x \cos 2x$ 5

$1 - \cos^2 2x + \cos 2x = 1 - (1 - \cos^2 2x)\cos 2x$ 5

$\cos^3 2x + \cos^2 2x - 2\cos 2x = 0$ 5

$t(t^2 + t - 2) = 0$

$t = 0$ 5 $\quad \left(t + \frac{1}{2}\right)^2 - 2 - \frac{1}{4} = 0 \quad \Rightarrow \quad t = 1$

$t = -2$ ✗ 5

$\cos 2x = \cos\frac{\pi}{2}$ 5 $\qquad \cos 2x = \cos 0$

$2x = 2n\pi \pm \frac{\pi}{2}\,;\ n \in \mathbb{Z} \qquad 2x = 2m\pi \quad m \in \mathbb{Z}$

$x = n\pi \pm \frac{\pi}{4}$ 5 $\qquad x = m\pi$ 5

$\boxed{50}$